

Nonequilibrium quench dynamics in quantum quasicrystals

Ferenc Igloi^{1,2}, Gergő Roósz^{2,1}

¹ Wigner Research Centre, Institute for Solid State Physics and Optics, H-1525 Budapest, P.O.Box 49, Hungary

² Institute of Theoretical Physics, Szeged University, H-6720 Szeged, Hungary

E-mail: igloi.ferenc@wigner.mta.hu

E-mail: gergo-roosz@titan.physx.u-szeged.hu

Yu-Cheng Lin

Graduate Institute of Applied Physics, National Chengchi University, Taipei, Taiwan

E-mail: yc.lin@nccu.edu.tw

Abstract. We study the nonequilibrium dynamics of a quasiperiodic quantum Ising chain after a sudden change in the strength of the transverse field at zero temperature. In particular we consider the dynamics of the entanglement entropy and the relaxation of the magnetization. The entanglement entropy increases with time as a power-law, and the magnetization is found to exhibit stretched-exponential relaxation. These behaviors are explained in terms of anomalously diffusing quasiparticles, which are studied in a wave packet approach. The nonequilibrium magnetization is shown to have a dynamical phase transition.

1. Introduction

Recent experimental progress in ultracold atomic gases in optical lattices [1, 2, 3, 6, 5, 4, 7, 8, 9, 10] has opened fascinating new perspectives on research in the field of isolated quantum systems, both in equilibrium and out of equilibrium. In experiments the form of atomic interactions can be suddenly changed by tuning an applied magnetic field near a Feshbach resonance, which is known as a global quantum quench. On the theoretical side, one is interested in the time-evolution of different observables, such as the order parameter or some correlation function, after a quench. Fundamental questions concerning quantum quenches include (i) the functional form of the relaxation process in early times, and (ii) the properties of the stationary state of the system after a sufficiently long time.

Many results for quantum quenches have been obtained for *homogeneous* systems [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32]; for example, the relaxation of correlation functions in space and in time is generally in an exponential form, which defines a quench-dependent correlation length and a relaxation (or decoherence) time. Many basic features of the relaxation process can be successfully explained by a quasiparticle picture [33, 14, 34]: after a global quench quasiparticles are created homogeneously in the sample and move ballistically with momentum dependent velocities. The behavior of observables in the stationary state is generally different in integrable and in non-integrable systems. For non-integrable models, thermalization is expected [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22] and the distribution of an observable is given by a thermal Gibbs ensemble; however, in some specific examples this issue has turned out to be more complex [23, 24, 25, 31]. By contrast, it was conjectured that stationary state averages for integrable models are described by a generalized Gibbs ensemble [12], in which each integral of motion is separately associated with an effective temperature.

Concerning quantum quenches in inhomogeneous systems, there have been only a few studies in specific cases; for example, entanglement entropy dynamics in random quantum chains [35, 36, 37] and in models of many-body localization [38, 39]. In some of these cases the eigenstates are localized, which prevents the system from reaching a thermal stationary state.

A special type of inhomogeneity, interpolating between homogeneous and disordered systems, is a quasicrystal [40, 41] or an aperiodic tiling [42]. Quasicrystals are known to have anomalous transport properties [43, 44], which is due to the fact that in these systems the long-time motion of electrons is not ballistic, but an anomalous diffusion described by a power law. One may expect that the quasiparticles created during the quench have similar dynamical behavior, which in turn affects the relaxation properties of quasicrystals.

Quasicrystals of ultracold atomic gases have been experimentally realized in optical lattices by superimposing two periodic optical waves with different incommensurate wavelengths. An optical lattice produced in this way realizes a Harper's quasiperiodic potential [47, 48], for which the eigenstates are known to be either extended or localized depending on the strength of the potential. Different phases of Bose-Hubbard model with such a potential have been experimentally investigated [45, 46]. There have been also theoretical studies concerning the relaxation process in the Harper potential [49, 50].

In this paper we consider the nonequilibrium quench dynamics of the quantum Ising chain in one-dimensional quasicrystals. The quantum Ising chain in its

homogeneous version is perhaps the most studied model for nonequilibrium relaxation [51, 52, 53, 54, 55, 56, 57, 58, 60, 59, 34, 61, 62, 63, 64, 65, 66]. Our study extends previous investigations in several respects and seek to obtain new insights into quench dynamics in inhomogeneous systems. We focus on the Fibonacci lattice, for which many equilibrium properties of the quantum Ising model are known [67, 68, 69, 70, 71, 72, 73]; to our knowledge this is the first study of quantum quenches in such a lattice. Using free-fermionic techniques [74], we numerically calculate the time-dependence of the entanglement entropy as well as the relaxation of the local magnetization for large lattices. The numerical results are interpreted by a modified quasiparticle picture, in which the quasiparticles are represented by wave packets; we also obtain diffusive properties of the wave packets.

The structure of the paper is as follows. The quasiperiodic quantum Ising model and its equilibrium properties are described in section 2. The global quench process and some known results for homogeneous and random chains are presented in section 3. Our numerical results for the quasiperiodic chain are presented and interpreted in section 4. This paper is closed by a discussion; some details of the free-fermionic calculation of the local magnetization are presented in the appendix.

2. The Model and its equilibrium properties

We consider the quantum (or transverse) Ising model defined by the Hamiltonian:

$$\mathcal{H} = -\frac{1}{2} \left[\sum_i J_i \sigma_i^x \sigma_{i+1}^x - h \sum_i \sigma_i^z \right], \quad (1)$$

where σ_i^x and σ_i^z are Pauli matrices at site i . The interactions, J_i , are generally site dependent, which are parameterized as:

$$J_i = J r^{f_i}, \quad (2)$$

where $r > 0$ is the amplitude of the inhomogeneity, and the integers f_i are taken from a quasiperiodic sequence.

Quasiperiodic lattices can be generated in different ways, such as by the cut-and-project method. Here we use the following algebraic definition for a one-dimensional quasiperiodic sequence:

$$f_i = 1 + \left[\frac{i}{\omega} \right] - \left[\frac{i+1}{\omega} \right], \quad (3)$$

where $[x]$ denotes the integer part of x , and $\omega > 1$ is an irrational number. The Fibonacci sequence generated by the substitution rule: $0 \rightarrow 01$ and $1 \rightarrow 0$ starting with 0 corresponds to the formula in (3) with the golden mean $\omega = (\sqrt{5} + 1)/2$. The parameter J in (2) is fixed with $J = r^{-\rho}$, where

$$\rho = \lim_{L \rightarrow \infty} \frac{\sum_{i=1}^L f_i}{L} = 1 - \frac{1}{\omega}, \quad (4)$$

is the fraction of units 1 in the infinite sequence. Note that $r = 1$ represents the homogeneous lattice.

The model in (1) can be transformed into a free fermionic problem [74] (see Appendix). The spectrum of free-fermionic excitations, ϵ_q in (A.1), plays a key role in equilibrium and non-equilibrium properties of the system. In equilibrium and in the thermodynamic limit the model has a quantum critical point at $h = h_c$, the properties

of which are controlled by the low-energy excitations. The value of h_c is determined by the equation [75] $\ln h_c = \overline{\ln J}$, where the overbar denotes an average over all sites. With the parameterization given above, the critical point is given by $h_c = 1$, independently of r . The lowest gap, $\Delta E = \epsilon_1$, is zero for $h < h_c$, and it vanishes as $\Delta E \sim (h_c - h)^\nu$, as h approaches h_c . The singularity of the gap, measured by the gap-exponent $\nu = 1$, does not depend on r ; the same is true for the singularity of the specific heat: $C_v \sim \ln |h - h_c|$. Thus the transition belongs to the Onsager universality class irrespectively of r . This means that the quasiperiodic modulation of the couplings represents an irrelevant perturbation at the critical point of the homogeneous model [76]. For $h < h_c$ the system is in the ordered phase, so that the local magnetization at site l is $m_l > 0$. The bulk (surface) magnetization vanishes at the critical point as $m_b(h) \sim (h_c - h)^{1/8}$ ($m_1(h) \sim (h_c - h)^{1/2}$). For $h > h_c$ the system is in the disordered phase and the local magnetization vanishes in the thermodynamic limit.

While in equilibrium only the low-energy excitations are of importance, the complete energy spectrum contributes to nonequilibrium properties, which are investigated in this paper.

3. Nonequilibrium properties of homogeneous and random chains

We consider a quench process in which at time $t = 0$ the strength of the transverse field is changed suddenly from h_0 to another value, say h . The initial Hamiltonian with h_0 for $t < 0$ is denoted by \mathcal{H}_0 , and its ground state is $|\Psi_0^{(0)}\rangle$. For $t > 0$ the new Hamiltonian \mathcal{H} with h governs the coherent time-evolution of the system; for example an observable, represented by the operator \hat{A} , has the time-evolution in the Heisenberg picture as: $\hat{A}(t) = \exp(it\mathcal{H})\hat{A}\exp(-it\mathcal{H})$, and its expectation value for $t > 0$ is given by $A(t) = \langle \Psi_0^{(0)} | \hat{A}(t) | \Psi_0^{(0)} \rangle$. Dynamics of the system out-of-equilibrium is governed by the complete spectrum of \mathcal{H} and not only by the lowest excitations. Therefore, Hamiltonians with different spectral properties will have completely different nonequilibrium properties.

The form of the inhomogeneity in the couplings is generally crucial to the spectrum of a Hamiltonian. For example the spectrum of the homogeneous quantum Ising chain is absolutely continuous, thus all the eigenstates are extended. By contrast, the random chain has a singular point spectrum and the eigenstates are localized. The spectrum of quasiperiodic chains lies between the above mentioned two limiting cases [77, 78]; for example, the spectrum of the Fibonacci chain defined in (1) is given by a Cantor set of zero Lebesgue measure, thus purely singular continuous [79].

Below we first briefly review nonequilibrium properties of the entanglement entropy and local magnetization after a quench in the homogeneous chain and in random chains.

3.1. Entanglement entropy

The *entanglement entropy*, $\mathcal{S}_\ell(t)$, of a block of the first ℓ sites in the chain is defined as $\mathcal{S}_\ell(t) = -\text{Tr}_\ell[\rho_\ell(t) \ln \rho_\ell(t)]$ in terms of the reduced density matrix: $\rho_\ell(t) = \text{Tr}_{i>\ell} |\Psi_0(t)\rangle \langle \Psi_0(t)|$. Here $|\Psi_0(t)\rangle$ denotes the ground state of the complete system at time $t > 0$. Details of the calculation of $\mathcal{S}_\ell(t)$ in the free-fermion representation can be found in the appendix of [80].

For the *homogeneous chain* (corresponding to the case with $r = 1$ in (2)) in the limit $L \rightarrow \infty$ and for $\ell \gg 1$ the results can be summarized as follows [33, 36]:

$$S_\ell(t) = \begin{cases} \alpha t, & t < \ell/v_{\max} \\ \beta \ell, & t \gg \ell/v_{\max}, \end{cases} \quad (5)$$

where v_{\max} is a maximum velocity. For a quench to a quantum critical point, the result in (5) is a consequence of conformal invariance [33]; for other cases, this behavior can be explained in the frame of a semiclassical (SC) theory [33, 34]: entanglement between the subsystem and its environment arises when two quantum entangled quasiparticles, which are emitted at $t = 0$ and move ballistically with opposite velocities, arrive in the subsystem and in the environment simultaneously. The prefactors $\alpha = \alpha(h_0, h)$ and $\beta = \beta(h_0, h)$ have been exactly calculated [81] and these agree with the results obtained from the SC theory [34]. In [82] $\alpha(h_0 = 0, h)$ has been evaluated in a closed formula, which is a continuous function of h but at the critical point $h = 1$, its second derivative is logarithmically divergent.

In the *random chain* the excitations are localized and therefore the dynamical entanglement entropy approaches a finite limiting value. When the quench is performed to the random quantum critical point, the average entropy increases ultra-slowly as $\log \log t$ [36]. This behavior can be explained in terms of the strong disorder renormalization group [83, 84, 36, 39].

3.2. Local magnetization

Another quantity we consider is the *local magnetization*, $m_l(t)$, at a position, l , of an open chain. Following Yang [85] this is defined for large L as the off-diagonal matrix-element:

$$m_l(t) = \left\langle \Psi_0^{(0)} | \sigma_l^x(t) | \Psi_1^{(0)} \right\rangle, \quad (6)$$

where $|\Psi_1^{(0)}\rangle$ is the first excited state of the initial Hamiltonian. Calculation of the magnetization in terms of free fermions is outlined in Appendix.

For the *homogeneous chain* the time-dependence of the local magnetization has been numerically calculated in [34, 58]. For the quench performed within the ordered phases, $h_0 < 1$ and $h < 1$, the results in the limit $L \rightarrow \infty$ and $l \gg 1$ are given by:

$$m_l(t) \sim \begin{cases} \exp(-t/\tau), & t < l/v_{\max} \\ \exp(-l/\xi), & t \gg l/v_{\max}, \end{cases} \quad (7)$$

where the relaxation (decoherence) time τ and the correlation length ξ depend on the quench parameters h_0 and h . Exact expressions of these quantities have been derived recently [61, 63, 64]. In the small h_0 and h limit, accurate results can also be obtained from the SC theory [34, 86]. In this framework the quasiparticles in terms of the σ operators are represented by ballistically moving kinks. Each time when a kink passes site l , the σ_l^z operator changes sign; thus kinks that pass a site an even number of times have no effect on the local magnetization. Summing up contributions of all kinks, we obtain the functional form in (7). If the quench is performed close to the critical point, the kinks have a finite width; this effect can be taken into account in a modified SC theory [34, 65], which provides exact results.

For quenches involving the disordered phase with $h_0 > 1$ and/or $h > 1$, the results obtained numerically [34, 58] or analytically by the form-factor approach [61, 63, 64] indicate that the local magnetization changes sign during the relaxation process. The

period of these oscillations increases and becomes divergent as $h \rightarrow 1^+$, signaling a *dynamical phase-transition* in the system.

In an *disordered chain* away from the random quantum critical point the bulk magnetization approaches a finite limiting value, which reflects the localized nature of the excitations. After a quench performed to the critical point, the average bulk magnetization has been found to vanish asymptotically in a very slow way [87], $m_b(t) \sim \ln^{-A}(t)$, where $A > 0$ is a disorder dependent constant.

4. Results for quasiperiodic chains

In this section we present our results for the quasiperiodic quantum Ising chain after a global quench, obtained by numerical calculations based on the free-fermion representation of the model. We concentrate on the Fibonacci chain with the parameter ω defined in (3) being the golden mean. We consider finite chains with a length fixed to a Fibonacci number F_n . We have calculated the entanglement entropy and the local magnetization for system sizes up to $L = F_{17} = 1597$. Below we present results for these two quantities separately.

4.1. Entanglement entropy

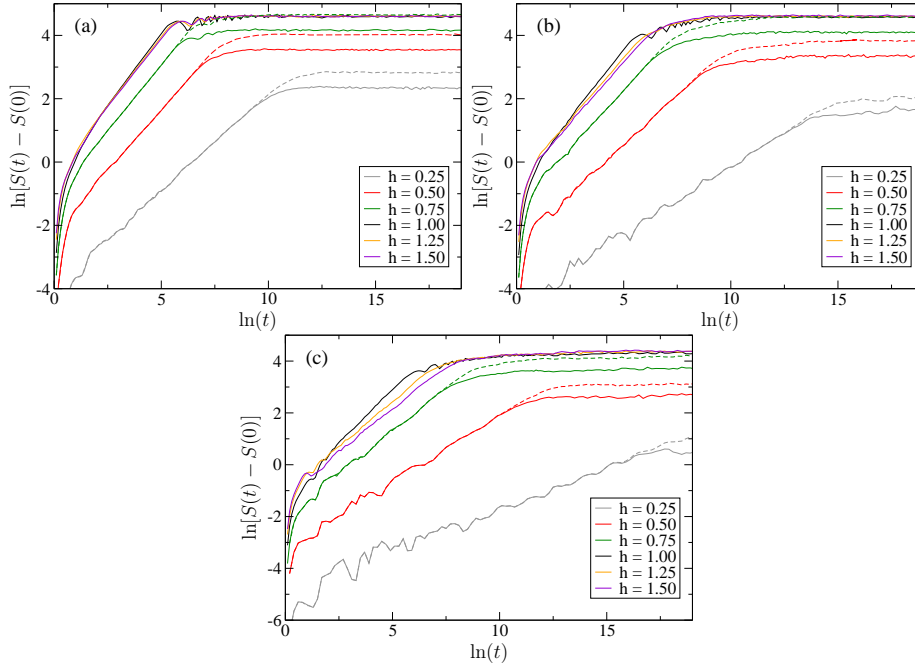


Figure 1. Dynamical entropy after a quench from $h_0 = 0$ to various values of h at the aperiodicity parameters (a) $r = 0.75$, (b) $r = 0.5$ and (c) $r = 0.25$. The solid lines are the results for $L = F_{16} = 987$, and the dashed lines (only at $h = 0.25$, $h = 0.5$ and $h = 0.75$) correspond to the data for $L = F_{17} = 1597$.

For a chain of total length F_n with periodic boundary conditions, we have calculated the entanglement entropy S_ℓ between a block of length $\ell = F_{n-2}$ and

its environment which has a length of F_{n-1} . Various values of $0 < r < 1$ for the inhomogeneity amplitude were considered. We start our numerical calculations from the fully ordered state with $h_0 = 0$ to a state with $h > 0$ both in the ordered and in the disordered phases, as well as at the critical point. The numerical results for $S_\ell(t) - S_\ell(0)$ are shown in figure 1. For all cases considered, $S_\ell(t)$ exhibits two time-regimes: in the late-time regime, the entropy is saturated to an L dependent value, similar to the behavior for the homogeneous chain; in the early-time regime, it increases with time as a power-law form:

$$S(t) \sim t^\sigma, \quad (8)$$

with some exponent $\sigma < 1$. The exponent σ depends on the value of the transverse field in the final state, while it does not vary (significantly) with the initial h_0 . The values of σ for $r = 0.25, 0.5$ and 0.75 are plotted in figure 5; for all cases considered, σ reaches its maximum at the critical point $h = 1$, and the increase with h in the ordered phase ($h < 1$) is much faster than the decrease in the disordered phase ($h > 1$). Furthermore, we have found that the exponent σ decreases with stronger inhomogeneity, that is with smaller value of r .

The power-law time-dependence of the entanglement entropy in (8) is a new feature of the quasiperiodic system: the increase in entropy is slower than in the homogeneous chain, but faster than in a random chain. This behavior can be explained in terms of quasiparticles that are emitted at time $t = 0$, and subsequently move classically by anomalous diffusion which has a power-law relationship between displacement and time, $x \sim t^D$, with a diffusion exponent $0 < D < 1$. The dynamics of the quasiparticles will be studied in more detail in section 4.3.

4.2. Local magnetization

The local magnetization, $m_l(t)$, is calculated for open chains of length $L = F_n$. Generally $m_l(t)$ has a monotonous position dependence: $m_{l_1}(t) > m_{l_2}(t)$ for $l_1 < l_2 < L/2$. We measured the magnetization at site $l = F_{n-1}$, which is considered as the bulk magnetization and denoted by $m_b(t)$. We have also studied the behavior of the surface magnetization, $m_1(t)$, for which some exact results are obtained.

We study the asymptotic behavior of the surface magnetization (given in (A.17)) for large t after a quench. If the quench is performed to the ordered phase, $h < 1$, the lowest excitation energy is $\epsilon_1 = 0$ (i.e. $\cos(\epsilon_1 t) = 1$); consequently $P_{1,2k-1}(t)$ in (A.8) has a time independent part. This results in non-oscillating contribution to the surface magnetization by

$$\overline{m_1} = \Phi_1(1) \sum_{j=1}^L \Phi_1(j) \Phi_1^{(0)}(j), \quad (9)$$

which corresponds to its asymptotic value. Recall that $\Phi_1(1) = m_1(h, t = 0)$ is the equilibrium surface magnetization [88, 89], which is finite for $h < 1$, and zero in the disordered phase. Similarly, $\Phi_1^{(0)}(1) > 0$ for $h_0 < 1$ and zero otherwise. From this follows that the nonequilibrium surface magnetization is $\overline{m_1} > 0$, if both $h < 1$ and $h_0 < 1$. Otherwise the stationary surface magnetization vanishes. If the quench starts from the fully ordered initial state $h_0 = 0$, then $\Phi_1(j) = \delta_{1,j}$ and $\overline{m_1} = \Phi_1^2(1)$; thus we obtain the simple relation:

$$\overline{m_1}(h) = [m_1(h, t = 0)]^2, \quad (10)$$

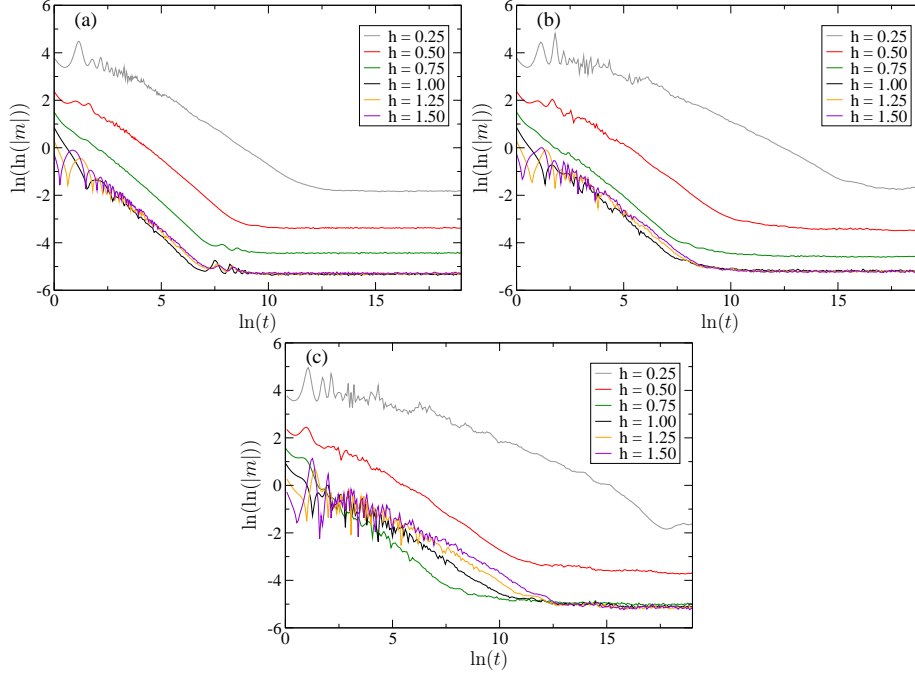


Figure 2. Double logarithm of the bulk magnetization as a function of the logarithm of the time. During the quench the transverse field is changed from $h_0 = 0$ to different values of h at the aperiodicity parameter $r = 0.75$ (panel (a)), $r = 0.5$ (panel (b)), $r = 0.25$ (panel (c)). The length of the chain is $L = F_{17} = 1597$ and the magnetization is considered at site $l = F_{16} = 987$.

which is generally valid between the stationary value of the nonequilibrium surface magnetization and its equilibrium value. From (10) follows that the critical exponents are related as: $\beta_s^{\text{ne}} = 2\beta_s$. For the Fibonacci chain there is a linear variation close to the critical point: $\overline{m}_1(h) \sim 1 - h^2$, thus $\beta_s^{\text{ne}} = 1$.

We numerically calculated the time-dependence of the bulk magnetization after a quench from the fully ordered initial state, $h_0 = 0$, to different values of h . For fixed values of the inhomogeneity $r = 0.25, 0.5, 0.75$, the results for the double logarithm of $|m_b(t)|$ are shown in figure 2 as functions of $\ln t$. In each case one can observe a linear dependence, which implies that the magnetization has asymptotically a stretched exponential time dependence:

$$m_b(t) \sim A(t) \exp(-Ct^\mu) . \quad (11)$$

Here the prefactor $A(t)$ changes its functional form at some critical value, $h^*(r)$, of the transverse field. For small transverse fields $h < h^*(r)$, the magnetization stays positive with $A(t) > 0$ during the process. For larger transverse fields $h > h^*(r)$, $A(t)$ changes sign in an approximately periodic fashion; thus in this region the magnetization oscillates in time; the oscillations are characterized by a typical time period, denoted by $t^*(h, r)$. As some critical value, $h^*(r)^+$, of the transverse field is approached, $h \rightarrow h^*(r)^+$, $t^*(h, r)$ becomes divergent. Thus at $h^*(r)$ there is a dynamical phase transition, similar to that noticed in the homogeneous model at $h_c = 1$. Interestingly, for the quasiperiodic model this dynamical phase transition does not coincide with

the equilibrium phase transition. Estimates of $h^*(r)$ versus r are shown in figure 3; the data are well approximated by a power-law $h^*(r) \sim r^\alpha$ with $\alpha = 0.24(3)$ [90].

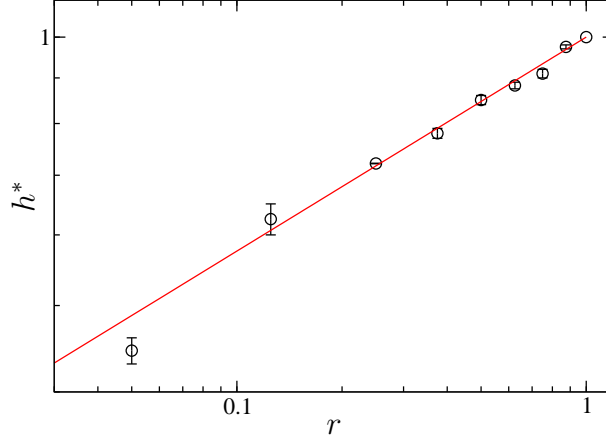


Figure 3. Position of the dynamical critical point for different values of the aperiodicity parameter in a double-logarithmic plot. The straight line has a slope $\alpha = 0.24$.

The exponent μ describing the decay of the local magnetization depends both on h and r ; by contrast, it does not vary significantly with h_0 , at least for $h_0 < h$. Our results for the critical exponents $\mu = \mu(h, r)$ are plotted in figure 5 for $r = 0.75$, 0.5 and 0.25 as functions of h . The exponent μ reaches its maximum at the dynamical phase transition point $h^*(r)$.

4.3. Interpretation by wave packet dynamics

As known from previous studies on the homogeneous chain, dynamical features of the entanglement entropy and the local magnetization can be well described by the dynamics of quasiparticles. To understand the dynamical properties of the quasiparticles emitted after a quantum quench in the quasiperiodic lattice, we regard the quasiparticles as wave packets and study their dynamics using a method that has been applied for studies of transport properties of quasicrystals [44, 91].

First we notice that the time dependencies of the fermion operators for the quantum Ising chain are given in (A.7) and (A.8). To construct a wave packet, which connects sites k and l at time t , we combine the four time-dependent factors in (A.8) in the form:

$$W_{l,k}(t) = \frac{1}{2} [P_{2l-1,2k-1} + P_{2l,2k} + i(P_{2l-1,2k} - P_{2l,2k-1})] , \quad (12)$$

which is localized at $t = 0$ since $W_{l,k}(0) = \delta_{l,k}$. The width of the wave packet starting from site k after time t is given by:

$$d(k, t) = \left[\sum_l (k-l)^2 |W_{l,k}(t)|^2 \right]^{\frac{1}{2}} . \quad (13)$$

One expects that for large t the wave packet in the infinite system shows anomalous diffusion in the form $d(k, t) \sim t^{D(k)}$ with a diffusion exponent $D(k)$, which generally

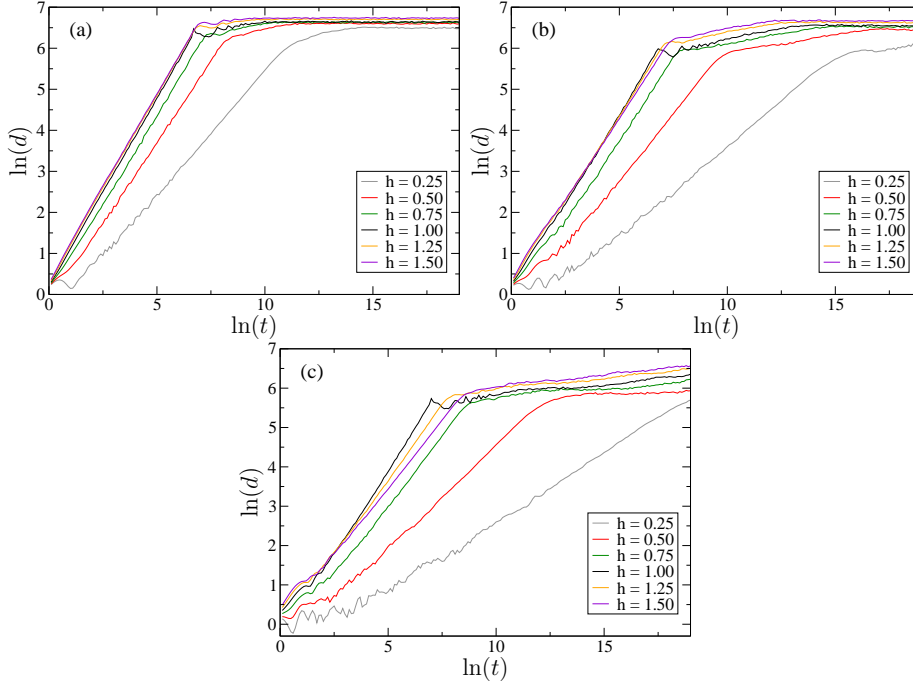


Figure 4. Time-dependent width of the wave-packet at different values of h for $r = 0.75$ (panel (a)), $r = 0.5$ (panel (b)), $r = 0.25$ (panel (c)).

depends on the starting position. After a global quench, quasiparticles are emitted everywhere in lattices, therefore $d(k, t)$ should be averaged over different initial positions,

$$d(t) = \overline{d(k, t)} \sim t^D. \quad (14)$$

In our numerical calculations chains of length $L = F_{17} = 1597$ with periodic boundary conditions were considered. First we have confirmed that the wave packet constructed in our method moves ballistically in the homogeneous chain (with $r = 1$), corresponding to $D = 1$. In the quasiperiodic chains the motion is indeed anomalous diffusive with $D < 1$, which is seen in figure 4 where the average widths of the wave packet are presented as functions of time in a log-log plot for various values of h and $r = 0.75, 0.5$ and 0.25 . The diffusion exponent D for given h and r corresponds to the slope of the linear part of the function.

The variation of D with h at a fixed r is shown in figure 5, compared with the exponent σ for the entanglement entropy and the exponent μ for the local magnetization. Here one can observe that the agreement between these three exponents is very good for $h < h^*(r)$, i.e. in the non-oscillating phase, but the exponent for the magnetization deviates in the oscillating phase ($h > h^*(r)$). The discrepancy in the oscillating phase implies that the semiclassical picture breaks down in the oscillating phase, where the quasiparticles cannot well described by the moving kinks in the magnetization.

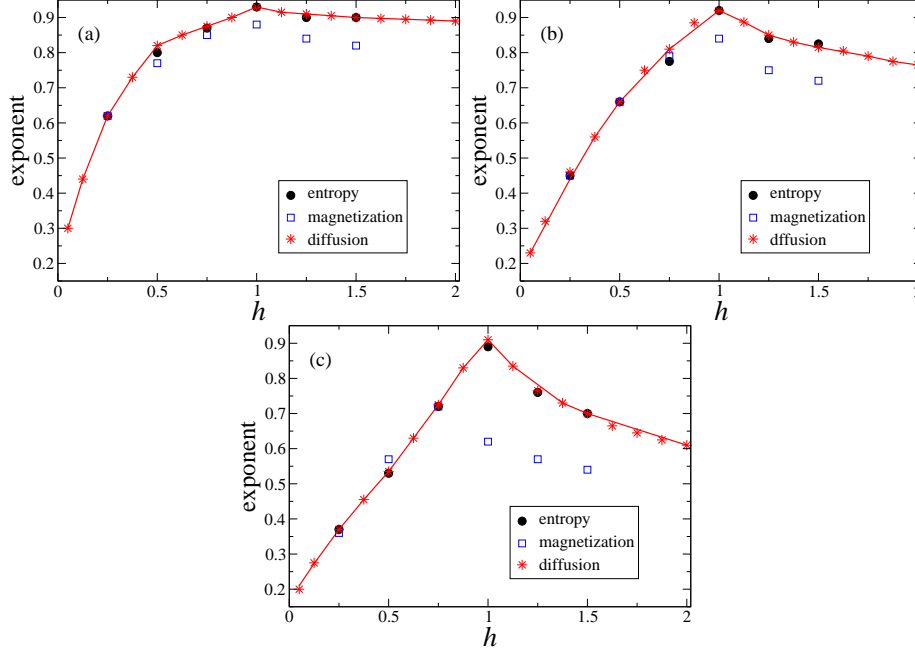


Figure 5. Scaling exponents calculated from the time-dependence of the width of the wave-packet, from the entanglement entropy and from the magnetization at different values of h for $r = 0.75$ (panel (a)), $r = 0.5$ (panel (b)), $r = 0.25$ (panel (c)). The full lines connecting the diffusion exponents are guides for the eye.

5. Discussion

In this paper we have studied the nonequilibrium dynamics of quasiperiodic quantum Ising chains after a global quench. In a quench process, the complete spectrum of the Hamiltonian is relevant for the time evolution of various observables. For the quasiperiodic quantum Ising chain the spectrum is in a very special form, which is given by a Cantor set of zero Lebesgue measure, thus it is purely singular continuous. We have calculated numerically two quantities: the dynamical entanglement entropy and the relaxation of the local magnetization. The entanglement entropy is found to increase in time as a power-law (see (8)), whereas the bulk magnetization decays in a stretched exponential way (see (11)). Both behaviors can be explained in a quasiparticle picture, in which the quasiparticles move by an anomalous diffusion in the quasiperiodic lattice. The diffusion exponent has been calculated by a wave packet approach, and good agreement has been found with the exponents that we obtained for the entropy and for the magnetization. We note that the anomalous dynamics found in the global quench process is similar to the transport properties of quasicrystals.

Relaxation of the bulk magnetization is found to present a nonequilibrium dynamical phase transition. The non-oscillating phase, in which the magnetization is always positive, and the oscillating phase, in which the sign of the magnetization varies periodically in time, is separated by a dynamical phase transition point, at which the time-scale of oscillations diverges. This singularity point, due to collective

dynamical effects, is different from the equilibrium critical point.

Similar nonequilibrium dynamical behavior is expected to hold for other quasiperiodic or aperiodic quantum models as long as the spectrum of the Hamiltonian is also purely singular continuous; there is a large class of such models, for example the Thue-Morse quantum Ising chain. If, however the spectrum of the Hamiltonian of the model is in a different type, such as the Harper potential which has extended or localized states, the nonequilibrium dynamics is expected to be different than the case we consider in this paper.

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Appendix A. Free-fermionic calculation of the time-dependent local magnetization

Using the Jordan-Wigner transformation the Hamiltonian in (1) can be written in a quadratic form in fermion operators, which is diagonalized through a canonical transformation, yielding

$$\mathcal{H} = \sum_{k=1}^L \epsilon_k \left(\eta_k^\dagger \eta_k - \frac{1}{2} \right). \quad (\text{A.1})$$

Here the energies of free fermionic modes, ϵ_k , are given by the solutions of the equations:

$$\begin{aligned} \epsilon_k \Psi_k(l) &= -h \Phi_k(l) - J_k \Phi_k(l+1), \\ \epsilon_k \Phi_k(l) &= -J_{k-1} \Psi_k(l-1) - h \Psi_k(l) \end{aligned}$$

with normalized vectors Ψ_k and Φ_k . For the calculation of the local magnetization we use free boundary conditions with $J_L = 0$, corresponding to $\Phi_k(L+1) = \Psi_k(0) = 0$.

To calculate of the local magnetization in (6), we introduce at each site two Majorana fermion operators, \check{a}_{2l-1} and \check{a}_{2l} , defined as

$$\begin{aligned} \check{a}_{2l-1} &= \sum_{k=1}^L \Phi_k(l) (\eta_k^\dagger + \eta_k), \\ \check{a}_{2l} &= -i \sum_{k=1}^L \Psi_k(l) (\eta_k^\dagger - \eta_k). \end{aligned} \quad (\text{A.2})$$

These satisfy the commutation relations:

$$\check{a}_l^+ = \check{a}_l, \quad \{\check{a}_l, \check{a}_k\} = 2\delta_{l,k}. \quad (\text{A.3})$$

The spin-operators are expressed in terms of the Majorana operators as:

$$\sigma_l^x = i^{l-1} \prod_{j=1}^{2l-1} \check{a}_j, \quad (\text{A.4})$$

and the local magnetization in (6) is then given as the expectation value of product of fermion operators with respect to the ground state.

$$m_l(t) = (\imath)^{l-1} \langle \Psi_0^{(0)} | \prod_{j=1}^{2l-1} \check{a}_j(t) \eta_1 | \Psi_0^{(0)} \rangle , \quad (\text{A.5})$$

where we have used: $|\Psi_1^{(0)}\rangle = \eta_1 |\Psi_0^{(0)}\rangle$. The expression in (A.5) - according to Wick's theorem - can be expressed as a sum of products of two-operator expectation values. This can be written in a compact form of a Pfaffian, which in turn can be evaluated as the square root of the determinant of an antisymmetric matrix:

$$m_l(t) = (-\imath)^{l-1} \left| \begin{array}{cccccc} \langle \check{a}_1(t) \check{a}_2(t) \rangle & \langle \check{a}_1(t) \check{a}_3(t) \rangle & \cdots & \langle \check{a}_1(t) \check{a}_{2l-1}(t) \rangle & \langle \check{a}_1(t) \eta_1 \rangle \\ & \langle \check{a}_2(t) \check{a}_3(t) \rangle & \cdots & \langle \check{a}_2(t) \check{a}_{2l-1}(t) \rangle & \langle \check{a}_2(t) \eta_1 \rangle \\ & & \ddots & & \vdots \\ & & & \langle \check{a}_{2l-2}(t) \check{a}_{2l-1}(t) \rangle & \langle \check{a}_{2l-2}(t) \eta_1 \rangle \\ & & & & \langle \check{a}_{2l-1}(t) \eta_1 \rangle \end{array} \right|$$

$$= \pm [\det C_{ij}]^{1/2} , \quad (\text{A.6})$$

where C_{ij} is the antisymmetric matrix $C_{ij} = -C_{ji}$, with the elements of the Pfaffian (A.6) above the diagonal. (Here and in the following we use the short-hand notation: $\langle \dots \rangle = \langle \Psi_0^{(0)} | \dots | \Psi_0^{(0)} \rangle$.)

The time evolution of the spin operators follows from the time dependence of the fermion operators: Inserting $\eta_k^\dagger(t) = e^{\imath t \epsilon_k} \eta_k^\dagger$ and $\eta_k(t) = e^{-\imath t \epsilon_k} \eta_k$ into (A.2) one obtains

$$\check{a}_m(t) = \sum_{n=1}^{2L} P_{m,n}(t) \check{a}_n , \quad (\text{A.7})$$

with

$$\begin{aligned} P_{2l-1, 2k-1} &= \sum_q \cos(\epsilon_q t) \Phi_q(l) \Phi_q(k), \\ P_{2l-1, 2k} &= - \sum_q \sin(\epsilon_q t) \Phi_q(l) \Psi_q(k) , \\ P_{2l, 2k-1} &= \sum_q \sin(\epsilon_q t) \Phi_q(k) \Psi_q(l) , \\ P_{2l, 2k} &= \sum_q \cos(\epsilon_q t) \Psi_q(l) \Psi_q(k) . \end{aligned} \quad (\text{A.8})$$

The two-operator expectation values are given by:

$$\langle \check{a}_m(t) \check{a}_n(t) \rangle = \sum_{k_1, k_2} P_{m, k_1}(t) P_{n, k_2}(t) \langle \check{a}_{k_1} \check{a}_{k_2} \rangle . \quad (\text{A.9})$$

The *equilibrium* correlations in the initial state with a transverse field h_0 are:

$$\begin{aligned} \langle \check{a}_{2m-1} \check{a}_{2n-1} \rangle &= \langle \check{a}_{2m} \check{a}_{2n} \rangle = \delta_{m,n}, \\ \langle \check{a}_{2m-1} \check{a}_{2n} \rangle &= -\langle \check{a}_{2m} \check{a}_{2n-1} \rangle = \imath G_{n,m}^{(0)} , \end{aligned} \quad (\text{A.10})$$

where the static correlation matrix $G_{m,n}^{(0)}$ is given by:

$$G_{m,n}^{(0)} = - \sum_q \Psi_q^{(0)}(m) \Phi_q^{(0)}(n) . \quad (\text{A.11})$$

Then (A.9) can be written in the form:

$$\langle \check{a}_m(t) \check{a}_n(t) \rangle = \delta_{m,n} + i\Gamma_{m,n}(t) , \quad (\text{A.12})$$

with

$$\begin{aligned} \Gamma_{2l-1,2m-1} &= \sum_{k_1,k_2} \left[G_{k_2,k_1}^{(0)} P_{2l-1,2k_1-1} P_{2m-1,2k_2} \right. \\ &\quad \left. - G_{k_1,k_2}^{(0)} P_{2l-1,2k_1} P_{2m-1,2k_2-1} \right] \\ \Gamma_{2l-1,2m} &= \sum_{k_1,k_2} \left[G_{k_2,k_1}^{(0)} P_{2l-1,2k_1-1} P_{2m,2k_2} \right. \\ &\quad \left. - G_{k_1,k_2}^{(0)} P_{2l-1,2k_1} P_{2m,2k_2-1} \right] \\ \Gamma_{2l,2m-1} &= - \sum_{k_1,k_2} \left[G_{k_2,k_1}^{(0)} P_{2l,2k_2} P_{2m-1,2k_1-1} \right. \\ &\quad \left. - G_{k_1,k_2}^{(0)} P_{2l,2k_2-1} P_{2m-1,2k_1} \right] \\ \Gamma_{2l,2m} &= \sum_{k_1,k_2} \left[G_{k_2,k_1}^{(0)} P_{2l,2k_1-1} P_{2m,2k_2} \right. \\ &\quad \left. - G_{k_1,k_2}^{(0)} P_{2l,2k_1} P_{2m,2k_2-1} \right] . \end{aligned} \quad (\text{A.13})$$

In (A.6) there are also the contractions:

$$\begin{aligned} \Pi_m &= \langle \Psi_0^{(0)} | \check{a}_m(t) \eta_1 | \Psi_0^{(0)} \rangle \\ &= \sum_n P_{m,n} \langle \Psi_0^{(0)} | \check{a}_n \eta_1 | \Psi_0^{(0)} \rangle \end{aligned} \quad (\text{A.14})$$

where

$$\begin{aligned} \langle \Psi_0^{(0)} | \check{a}_{2l-1} \eta_1 | \Psi_0^{(0)} \rangle &= \Phi_1^{(0)}(l) \\ \langle \Psi_0^{(0)} | \check{a}_{2l} \eta_1 | \Psi_0^{(0)} \rangle &= i\Psi_1^{(0)}(l) . \end{aligned} \quad (\text{A.15})$$

Thus finally the square of the local magnetization is given by the determinant:

$$m_l^2(t) = \begin{vmatrix} 0 & \Gamma_{1,2} & \Gamma_{1,3} & \cdots & \Gamma_{1,2l-1} & \Pi_1 \\ -\Gamma_{1,2} & 0 & \Gamma_{2,3} & \cdots & \Gamma_{2,2l-1} & \Pi_2 \\ -\Gamma_{1,3} & -\Gamma_{2,3} & 0 & \cdots & \Gamma_{3,2l-1} & \Pi_3 \\ & & & \ddots & & \vdots \\ -\Gamma_{1,2l-1} & \cdots & & & 0 & \Pi_{2l-1} \\ -\Pi_1 & \cdots & & & -\Pi_{2l-1} & 0 \end{vmatrix} \quad (\text{A.16})$$

As a special case, the surface magnetization is expressed as:

$$\begin{aligned} m_1(t) = \Pi_1 &= \sum_{j=1}^L P_{1,2j-1}(t) \Phi_1^{(0)}(j) \\ &\quad - i \sum_{j=1}^L P_{1,2j}(t) \Psi_1^{(0)}(j) . \end{aligned} \quad (\text{A.17})$$

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in figure 3 indicate that the critical value, $h^*(r)$ coincides with $r^{2/\omega-1}$.

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